

UDC 53.01

Dirac Equation on the Axially Symmetric Generalized Newman — Unti — Tamburino spacetime

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Abstract. We study the Dirac equation in the background of axially symmetric generalized Newman — Unti — Tamburino (NUT) spacetime. For massless particle, the problem is reduced to the one differential equation of second order and is solved in terms of Heun-type equation at special choice of arbitrary radial function.

Keywords: Dirac equation, generalized Newman — Unti — Tamburino space, spin $\frac{1}{2}$ particle, variable separation, massless case

Introduction. The axially symmetric metrics of the Taub-NUT type (so-called Euclidean Taub-NUT metric) were actively studied within grand unified theories as a monopole-like solutions [1]. In recent years, the original NUT metric acquires a fresh interest because it has been shown that the black holes with NUT charge can be considered as physically meaningful systems. In [2] the NUT charge effects has been assessed to be tested in the spectra of quasars, supernovae, or active galactic nuclei. The quantum-mechanical problem of particle in the background of black hole metrics is one of the interesting problem for cosmology and quantum gravity. The Dirac equation has been studied on the Kerr and Kerr — Newman spacetimes [3–4]. In [5] we found the solution of the Dirac equation for the massless particle in NUT background. The generalization of the Euclidean Taub-NUT metric has been proposed by Iwai and Katayama in [6]. They demonstrated that the proposed generalization accomplishes non-trivial properties. In this study we generalize the NUT-metric and study the Dirac equation in such axially symmetric background.

Methods and materials; results. We generalize Newman — Unti — Tamburino metric by introducing an arbitrary radial function $\Omega(r)$ in NUT line element:

$$ds^2 = \frac{\Delta}{\rho^2} \left(dt + 4a \sin^2 \theta / 2 d\phi \right)^2 - \frac{\rho^2}{\Delta} dr^2 - \frac{\rho^2}{\Omega(r)} \left(d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

$$\rho^2 = a^2 + r^2, \Delta = r^2 - 2Mr - a^2.$$

By employing the tetrad formalism, we derive the covariant Dirac equation for the massive particle as

$$\left[i \left(\gamma^0 \frac{\rho}{\sqrt{\Delta}} + \gamma^3 \frac{2a}{\rho} \sqrt{\Omega} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \right) \frac{\partial}{\partial t} - i \gamma^1 \left(\frac{\sqrt{\Delta}}{\rho} \frac{\partial}{\partial r} + \frac{r\sqrt{\Delta}}{2\rho^3} + \frac{\Delta'}{4\rho\sqrt{\Delta}} - \frac{\sqrt{\Delta}\Omega'}{2\rho\Omega} \right) + \right. \\ \left. + i \gamma^0 \gamma^2 \gamma^3 \frac{a\sqrt{\Delta}\Omega}{2\rho^3} - i \gamma^2 \frac{\sqrt{\Omega}}{\rho} \left(\frac{\partial}{\partial \theta} + \frac{1}{2\tan\theta} \right) - i \gamma^3 \frac{\sqrt{\Omega}}{\rho} \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} - M \right] \Psi = 0.$$

We separate the variables. The angular differential equations system for generalized metric is the same as for ordinary NUT metric. The four differential equations for radial functions R_1, R_2, R_3, R_4 read

$$\left(D_- - \frac{i\varepsilon\rho^2}{\sqrt{\Delta}} \right) R_3 + i\mu(r-ia)R_1 = \Lambda\sqrt{\Omega}R_4, \quad \left(D_- + \frac{i\varepsilon\rho^2}{\sqrt{\Delta}} \right) R_4 - i\mu(r-ia)R_2 = \Lambda\sqrt{\Omega}R_3, \\ \left(D_+ + \frac{i\varepsilon\rho^2}{\sqrt{\Delta}} \right) R_1 - i\mu(r+ia)R_3 = \Lambda\sqrt{\Omega}R_2, \quad \left(D_+ - \frac{i\varepsilon\rho^2}{\sqrt{\Delta}} \right) R_2 + i\mu(r+ia)R_4 = \Lambda\sqrt{\Omega}R_1,$$

with notation $D_{\pm} = \sqrt{\Delta} \frac{d}{dr} \pm \frac{i\sqrt{\Delta}}{2} \left(\frac{a(1+\Omega)}{\rho^2} \pm \frac{i\Omega'}{\Omega} \right)$. For massless case ($\mu = 0$) the

last system gives two independent systems of two equations.

Conclusion. The Dirac equation for massive particle is led to the differential equation of fourth order. The problem of massless spin $\frac{1}{2}$ particle is reduced to the one differential equation of second order, which is the Heun-type equation at special choice of radial function Ω .

References

- [1] Cotăescu I.I., Visinescu M. The Dirac field in Taub-NUT background. *Int. J. Mod. Phys.*, 2001, vol. 16 (10), pp.1743–1758.
- [2] Ghasemi-Nodehi M. Sgr A* shadow study with KTN space time and investigation of NUT charge existence. *Universe*, 2024, vol. 10, art. no. 378.
- [3] Chandrasekhar S. The Solution of Dirac's Equation in Kerr Geometry. *Proc. R. Soc. Lond. A*, 1976, vol. 349, pp. 571–575.
- [4] Dariescu C., Dariescu M.-A., Stelea C. Dirac equation on the Kerr — Newman spacetime and Heun Functions. *Advances in High Energy Phys.*, 2021, vol. 2021, art. no. 5512735.
- [5] Krylova N.G., Red'kov V.M. Dirac particle in gravitational field of black hole with Newman — Unti — Tamburino parameter. *Modern astronomy: from the Early Universe to exoplanets and black holes*, Proceedings of the VAK-2024 conference. SAO RAS, Nizhny Arkhyz, Russia, 2024. <https://doi.org/10.26119/VAK2024.013>
- [6] Iwai T., Katayama N. Two kinds of generalized Taub-NUT metria and the symmetry of associated dynamical systems. *J. Phys. A: Math. Gen.*, 1994, vol. 27, pp. 3179–3190.